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Boundary of a Boundary Principle and Geometric Structure of Field Theories

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We formulate the boundary of a boundary principle, review its applications in electrodynamics, Yang-Mills theory, and general relativity and translate its basic ideas into geometric language. In each of these three theories the density of the source lets itself be constructed—we discover—out of the curvature associated with the field as a Cartan-like moment of this curvature.

1. VANISHING OF THE BOUNDARY OF A BOUNDARY

No feature of physics more strongly supports the theme of "law without law" (Wheeler 1973, 1982)—or "physics and austerity"—than the automatic conservation of source as it shows itself in electrodynamics (ED), geometrodynamics (GMD) (Misner, *et al.*, 1973), and chromodynamics (ChD). Almost everything of all three theories lets itself be deduced from almost nothing. A mathematical identity, a tautology, a triviality, and yet the central principle of algebraic topology, the principle that the boundary of a boundary is zero (referred to hereafter as the BBP) (Spanier, 1966), dominates the architecture of each. For each that principle comes in twice over in determining the structure of the theory. Once the BBP comes in at the 1-2-3-dimensional level, in stating that the 1-dimensional boundary of a 2-dimensional boundary of a 3-dimensional region is zero; and again in the 2-3-4-dimensional form, according to which the 2-dimensional boundary of the 3-dimensional boundary of a 4-dimensional region is zero. The 1-2-3 BBP governs the structure of the field. The 2-3-4 BBP wires up the source to the field in such a way that the law of conservation of source is automatic, immediate, the consequence of a mathematical identity, not the doing of gears and pinions or Swiss watchmakers or writers of Lagrangians.

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The full parallelism between the application of the BBP in ED, GMD, and ChD has not been apparent in the past. In this paper the mathematics of these standard theories is restated in such a way as to exhibit this parallelism in a new light. Here for the first time it can be seen that the concept of "moment of rotation," introduced far back by Elie Cartan (1925, 1946) in the context of general relativity (GMD), and so sadly overlooked for so long, governs the coupling of field to source in all three theories.

The BBP lends itself to precise statement, including (1) the description of a chain complex associated with the manifold, (2) the boundary operator on the complex, and (3) the differential structure on the chain complex determined by the boundary operator. However, in this paper we are going to consider only local aspects of the principle. This means that only smooth, orientable, homotopically trivial manifolds with compact closure are going to be considered. It can be shown (Kheyfets, 1986) that in this case the consideration of the chain complex and the boundary operator can be replaced by the consideration of its dual cochain complex (de Rham complex of differential forms) and the operation of exterior derivative on differential forms. Thus the BBP can be expressed as the fact that the exterior derivative applied twice to any differential form gives zero. The proof of the eqivalence of this fact to the BBP for the given class of smooth manifolds is elementary and is based on Stokes' theorem (Kheyfets, 1986).

As in its integral formulation, so in its differential usage the BBP is used twice in all basic field theories—electrodynamics, general relativity, and the Yang-Mills theory (Misner *et al.*, 1973; Kheyfets and Wheeler, 1985). The first time the differential operator is applied twice to a 1-form, the resulting relation is called the 1-2-3 form of the BBP. The second time the differential operator is applied twice to a 2-form, the resulting relation is called the 2-3-4 form of the principle.

A look at the geometry of each field will recall how it is built and reveal how it uses the BBP principle.

2. ELECTRODYNAMICS AT FIRST SIGHT HAS NO PLACE FOR ANY "MOMENT OF ROTATION"

Electrodynamics, as is well known, can be considered as a geometry on the principal bundle with spacetime as its base and the structure group $U(1)$. This structure group is 1-dimensional and Abelian. Its Lie algebra is isomorphic as a vector space to the real axis. The connection on the principal bundle is determined by the 4-potential of the electromagnetic field. The electromagnetic field tensor is identified with the curvature of the connection, or, equivalently, with the gauge-covariant derivative of the connection form. Due to the fact that the group $U(1)$ is Abelian, the gauge-covariant derivative coincides with the exterior derivative. Also, because $U(1)$ is 1-dimensional, the connection form and the curvature form can be considered as scalar valued forms. For electrodynamics—contrary to the other two field theories—the curvature form of the geometry is gauge invariant. The gauge-covariant derivative is actually gauge invariant. In brief, in electrodynamics "covariance" usually means "invariance."

The BBP is applied in electrodynamics very simply. The connection 1-form (or 4-potential) is considered and the exterior derivative is applied to it twice. The result is an equation with the exterior derivative of the curvature form (or electromagnetic tensor) on the left and, according to the BBP, zero on the right. This relation expresses nothing but the homogeneous Maxwell equations. They represent, from the geometric point of view, the ordinary Bianchi identities for the curvature form. This concludes the description of the 1-2-3 form of the BBP in ED.

The homogeneous Maxwell equations do not determine the electromagnetic field completely. To complete the picture, the inhomogeneous Maxwell equations are required. To construct them, the Hodge star (duality) operator is applied to the electromagnetic form. The result is called the dual electromagnetic 2-form. The inhomogeneous Maxwell equations contain on the left the exterior derivative of the dual curvature form and the 4-current 3-form on the right. The exterior derivative can be applied to both sides of the inhomogeneous Maxwell equations. It follows then from the 2-3-4 form of the BBP (the exterior covariant derivative is applied twice to the dual electromagnetic 2-form) that the exterior derivative of the 4-current 3-form is equal to zero. This is nothing but the conservation law for the source of the electromagnetic field. Therefore, the 2-3-4 form of the boundary of a boundary principle provides the conservation law for the source of the electromagnetic field as an automatic consequence of the field equations.

It is interesting to notice here that the 1-2-3 form of the BBP in electrodynamics does not use the metric structure of spacetime, whereas the 2-3-4 form of the principle does, because the Hodge star operator is expressed in terms of the metric tensor.

But where is there any natural place in this description for any "moment of rotation"?

3. YANG–MILLS THEORY, COVARIANCE VERSUS INVARIANCE, AND AGAIN NO EVIDENT PLACE FOR ANY "MOMENT OF ROTATION"

It is no surprise that the geometric structure of the Yang-Mills theory and the content of the BBP in the theory can be described in a manner very similar to that in electrodynamics. The Yang–Mills theory can be considered as a geometry on the principal bundle with spacetime as the base and a Lie group as the structure group (the group of the internal symmetries). The structure group is usually assumed to be compact or semisimple, but, in general, it is neither 1-dimensional nor Abelian. The connection on this principal bundle is determined by the Yang–Mills potential, a 1-form with values in the Lie algebra of internal symmetry group. The strength of the Yang–Mills field is defined as the curvature 2-form of the connection [more exactly, as the pulldown (Bleecker, 1981) of the curvature 2-form to spacetime, but, since all our considerations are local, it is convenient to identify the curvature form with this pulldown], or, otherwise, as the gauge-covariant exterior derivative of the connection form. The curvature 2-form is horizontal (Bleecker, 1981). This circumstance allows one to extend the Hodge star operation to the form componentwise. It can be shown then (Kheyfets and Wheeler, 1985; Bleecker, 1981) that the ordinary Bianchi identities for the curvature 2-form are derived from the 1-2-3 form of the BBP and from the fact that the connection 1-form is vertical. These identities otherwise are called the homogeneous Yang–Mills equations. Exactly as in electrodynamics, they do not determine the Yang–Mills field completely. Required in addition are the inhomogeneous Yang–Mills equations. They contain the gauge-covariant exterior derivative of the dual curvature 2-form (Lie algebra valued) on the left and the current density 3-form (also Lie algebra valued) on the right. It has been shown [see Bleecker (1981) and also, for a more elementary proof, Kheyfets and Wheeler (1985)] that the second covariant derivative of the dual curvature 2-form vanishes, and that this follows from the 2-3-4 form of the boundary of a boundary principle and the circumstance that the connection form is vertical (Bleecker, 1981) and the curvature form is horizontal (Bleecker, 1981). As a consequence, application of the gauge-covariant exterior derivative to the inhomogeneous Yang–Mills equations automatically gives the conservation law for the source of the Yang–Mills field (the gauge-covariant exterior derivative of the source current 3-form is equal to zero).

In brief, both 1-2-3 and 2-3-4 forms of the BBP in the Yang–Mills theory work in almost exactly the same way as in electrodynamics, with the difference that gauge covariance shows up explicitly. The 1-2-3 form of the BBP is represented in both theories by ordinary Bianchi identities and has a very transparent geometric interpretation (Misner *et al.*, 1973; Bleecker, 1981). The 2-3-4 form of the principle gives the conservation law for the source current. However, the geometric interpretation of this relation in ED and ChD as a moment of rotation was unknown until recently (Kheyfets, 1986), even though the thought that it has to exist was discussed on many occasions (Wheeler, 1985).

4. CENTRAL PLACE OF MOMENT OF ROTATION IN GEOMETRODYNAMICS

This new view unites electrodynamics and Yang–Mills theory in form and spirit with general relativity. In all these cases the amount of field source within every small 3-dimensional region is expressed as the sum, over the boundary of that region, of the moment of rotation induced by the field on each element of this boundary. Irrelevant and canceling out in the final evaluation of the source, is the choice of the common point with respect to which those moments of rotation are calculated. This construction guarantees conservation of source as automatic consequence of the 2-3-4 BBP. This moment-of-rotation concept of a source and of automatic conservation of source is best introduced by recalling the concept of the moment of rotation as Elie Cartan introduced it in general relativity.

General relativity, as ED and ChD, can be considered as a geometry on a principal bundle with spacetime as the base. There is no full agreement on what has to be taken as the structure group of the bundle (Percacci, 1984). For our purpose—to discuss the BBP—it is convenient to take the group $GL(4)$ to be the structure group. This choice makes the principal bundle of general relativity the frame bundle of spacetime. The geometry itself is determined by a Levi-Civita connection 1-form (Lie algebra valued). This geometry is torsionless; that is, the covariant exterior derivative of the canonical soldering form $d\mathcal{P} = e_\mu dx^\mu$ (Bleecker, 1981) is equal to zero; and the covariant derivative of the metric tensor is equal to zero. It can be shown (Misner *et al.*, 1973; Bleecker, 1981) that the ordinary Bianchi identities are, just as in electrodynamics and the Yang–Mills theory, the consequence of the 1-2-3 BBP and the fact that the connection form of the geometry is, as always, vertical. The circumstance that the connection is a Levi-Civita connection has not been actively used so far.

The 2-3-4 form of the BBP principle, however, is considered in general relativity in a way ("moment of rotation") completely different, at first sight, from the way ED and ChD use that principle. Cartan—to put it in today's language—used the fact that the Lie algebra of the bundle can be reduced to the Lorentz group and can be made isomorphic (as a vector space) to the space of exterior 2-forms on a 4-dimensional space isomorphic to Minkowski space. Cartan introduced what would be called two Hodge star (duality) operators into general relativity. The right one acts on the scalar valued 2-forms. The left one acts on the Lie algebra. Lie algebra valued forms can be acted upon by both of them. Thus the concepts of left duality and right duality are distinguished. It was Cartan's great idea to introduce the moment of rotation associated with curvature (Cartan, 1946; Kheyfets and Wheeler, 1985). This moment of rotation can be written in modern notations as the left dual of the exterior product of the canonical soldering form and the curvature form. It can be shown, after some calculations (Misner *et al.*, 1973), that the moment of rotation is a vector valued (or 1-form valued) 3-form on spacetime and its components are equal to the components of the Einstein tensor. Essential here is the fact that the connection is a Levi-Civita connection. The Cartan moment of rotation is not well defined if the canonical soldering form is not covariantly constant; that is, if there is a nonzero torsion. It has been shown (Misner *et al.*, 1973) that the conservation of the moment of rotation follows from the 2-3-4 BBP. This conservation law states that the exterior covariant derivative of the Cartan moment of rotation is equal to zero or, equivalently, that the divergence of the Einstein tensor is equal to zero. The last relation leads automatically, via Einstein's equations, to conservation of the source of the gravitational field, that is, conservation of momentum-energy, in general relativity.

5. ED AND ChD VERSUS GMD. APPARENT STRUCTURAL DIFFERENCE TRANSLATED INTO UNITY VIA CONCEPT OF MOMENT OF ROTATION

The difference between general relativity on the one hand and electrodynamics and the Yang-Mills theory on the other hand is obvious. The 1-2-3 form of the BBP is treated in more or less the same way in all three theories. But the apparent difference in the treatment of the 2-3-4 form of the principle—and in the linkage of field to source—is objectionable. Indeed, the BBP is interesting first of all in the context of the idea of austerity—or law without law. However, a suspicion arises if the principle is applied completely differently in different basic field theories. Might not the whole content of each theory be contained in the way in which the theory used the 2-3-4 BBP principle? In that event the boundary of a boundary principle would hardly uphold the concept of austerity. There is a second concern. The geometric interpretation of the 2-3-4 BBP in terms of moment of rotation, so inspiring in general relativity, appears to be lost in electrodynamics and the Yang-Mills theory. With it is lost—it would seem—any unified use of the 2-3-4 BBP as means to incorporating source into a theory [cf., for example, Miller (1986)].

To achieve manifest unity it is necessary to have a universal recipe for constructing the source in a unified way in all basic field theories. The idea has been suggested (Wheeler, 1985) that the construction of the source as moment of rotation in general relativity can be extended to other basic field theories such as electrodynamics and the Yang-Mills theory. There are some difficulties, however. Recall that the moment is defined as the left dual of the exterior product of the canonical soldering form and the curvature form. But the principal bundles in electrodynamics and the Yang-Mills theory are not frame bundles, and, consequently, the canonical soldering form on them is not defined. The idea has been suggested (Kheyfets, 1986) to consider a theory which includes, along with gravitation, either electromagnetism or Yang-Mills theory or both, but still has a frame bundle as the underlying geometric structure.

For the case of electromagnetism, an obvious choice is the Kaluza-Klein theory in its 5-dimensional form. The calculation for it of the moment of rotation and its covariant exterior derivative has been performed. The result is that conservation of both momentum-energy and the source current automatically follow from the 2-3-4 form of the boundary of a boundary principle, and also the recipe for constructing the source current as a Cartan moment can be extracted. This recipe can be separated out so well that it can be transferred back to standard electrodynamics. The soldering form to be used in electrodynamics turns out to be just the canonical soldering form of the frame bundle over Minkowski space. Instead of the curvature form, the gradient of the electromagnetic tensor has to be used, the latter being considered (and it is hard to see why without reference to the Kaluza-Klein theory, but it is very natural in this theory) as a bivector-valued function on Minkowski space rather than a scalar-valued 2-form. If we take into account that the curvature form in general relativity describes the effects of the "gradient of gravitational force," we can say that in both electrodynamics and general relativity the source current density form can be expressed as Cartan's moment of the gradient of the field strength. The extension of this statement to the Yang-Mills theory turns out to be elementary and does not involve any new concepts (Kheyfets, 1985).

The 2-3-4 BBP in all three basic field theories—electrodynamics, the Yang-Mills theory, and general relativity—therefore can be expressed as the conservation law for Cartan's moment of the corresponding field strength gradient.

All three of the goals outlined above are thereby achieved: the correspondence between the BBP and the idea of austerity has been demonstrated, the unified geometric interpretation has been regained in all three theories, and a universal recipe for the incorporation of the source has been tested in all three basic field theories.

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